OLYMPIAD SOLUTIONS

OC111. Let x, y and z be positive real numbers. Show that

$$x^2 + xy^2 + xyz^2 \geq 4xyz - 4\,.$$

Originally question 1 from the 2012 Canadian Mathematical Olympiad.

Solved by A. Alt; Š. Arslanagić; M. Bataille; C. Curtis; R. Hess; D. Manes; P. Perfetti; and T. Zvonaru. We give the solution used by most of the solvers.

$$\begin{aligned} x^2 + xy^2 + xyz^2 - 4xyz + 4 &= x^2 - 4x + 4 + 4x - 4xy + xy^2 + 2xy + xyz^2 - 4xyz \\ &= (x-2)^2 + x(y-2)^2 + xy(z-2)^2 \geq 0. \end{aligned}$$

Editor's comment. Zvonaru remarked that we only used $x, y \geq 0$, but z could be any real number, which is obvious from the solution.

OC112. Find all pairs of natural numbers (a, b) that are not relatively prime ($\gcd(a, b) \neq 1$) such that

$$\gcd(a, b) + 9\,\mathrm{lcm}[a, b] + 9(a + b) = 7ab\,.$$

Originally question 4 from the Albanian team selection test, 2012.

Solved by A. Alt; Š. Arslanagić; M. Bataille; C. Curtis; and K. Zelator. We give the solution similar to the solutions of Šefket Arslanagić and Konstantine Zelator (done independently).

We claim that the only solutions are $(4, 38), (38, 4)$ and $(4, 4)$.

We will use the well known equality that $\gcd(a, b) \cdot \mathrm{lcm}[a, b] = a \cdot b$. The equation then becomes

$$\gcd(a, b) + 9\frac{ab}{\gcd(a, b)} + 9(a + b) = 7ab\,.$$

Let $d := \gcd(a, b)$ and write $a = dx, b = dy$, with $\gcd(x, y) = 1$. Then

$$\begin{aligned} d + \frac{9d^2xy}{d} + 9d(x + y) = 7d^2xy &\iff 1 + 9xy + 9(x + y) = 7dxy \\ &\iff d = \frac{1 + 9xy + 9(x + y)}{7xy}. \end{aligned}$$

Let us now observe that if $d \geq 5$ we have

$$\frac{1 + 9xy + 9(x + y)}{7xy} \geq 5 \iff 26xy \leq 1 + 9x + 9y.$$

But this is not possible, since $9x \leq 9xy$ and $9y \leq 9xy$ which would imply $8xy \leq 1$.